ON A MORELLI TYPE EXPRESSION OF COHOMOLOGY CLASSES OF TORIC VARIETIES

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ABSTRACT. Let X be a complete $\mathbb{Q}$-factorial toric variety of dimension n and Δ the fan in a lattice N associated to X. Δ is necessarily simplicial. For each cone σ of Δ there corresponds an orbit closure $V(\sigma)$ of the action of complex torus on X. The homology classes $\{[V(\sigma)] \mid \dim\sigma = k\}$ form a set of specified generators of $H_{n-k}(X,\mathbb{Q})$. It is shown that, given $\alpha \in H_{n-k}(X,\mathbb{Q})$, there is a canonical way to express α as a linear combination of the $[V(\sigma)]$ with coefficients in the field of rational functions of degree 0 on the Grassmann manifold $G_{n-k+1}(N_{\mathbb{Q}})$ of $(n-k+1)$-planes in $N_{\mathbb{Q}}$. This generalizes Morelli's formula [10] for α the $(n-k)$-th component of the Todd homology class of the variety X. Morelli's proof uses Baum-Bott's residue formula for holomorphic foliations applied to the action of complex torus on X whereas our proof is entirely combinatorial so that it tends to more general situations.

1. INTRODUCTION

Let X be a toric variety of dimension n and Δ_X the fan associated to X. Δ_X is a collection of rational convex cones in $N_{\mathbb{R}} = N\otimes\mathbb{R}$ where N is a lattice of rank n. For each k-dimensional cone σ in Δ_X, let $V(\sigma)$ be the corresponding orbit closure of dimension $n-k$ and $[V(\sigma)] \in A_{n-k}(X)$ be its Chow class. Then the Todd class $\mathscr{T}_{n-k}(X)$ of X can be written in the form

$$\mathscr{T}_{n-k}(X) = \sum_{\sigma\in\Delta_X,\dim\sigma=k} \mu_k(\sigma)[V(\sigma)]. \tag{1}$$

However, since the $[V(\sigma)]$ are not linealy independent, the coefficients $\mu_k(\sigma) \in \mathbb{Q}$ are not determined uniquely. Danilov [2] asks if $\mu_k(\sigma)$ can be chosen so that it depends only on the cone σ not depending on a particular fan in which it lies.

The equality (1) has a close connection with the number $\#(P)$ of lattice points contained in a convex lattice polytope P in $M_{\mathbb{R}}$ where M is a dual lattice of N. For a positive integer ν the number $\#(\nu P)$ is expanded as a polynomial in ν (called Ehrhart polynomial):

$$\#(\nu P) = \sum_k a_k(P)\nu^{n-k}.$$

A convex lattice polytope P in $M_{\mathbb{R}}$ determines a complete toric variety X and an invariant Cartier divisor D on X. There is a one-to-one correspondence between the cells $\{\sigma\}$ of Δ_X and the faces $\{P_\sigma\}$ of P. Then the coefficient $a_k(P)$ has an expression

$$a_k(P) = \sum_{\dim\sigma=k} \mu_k(\sigma)\operatorname{vol}P_\sigma \tag{2}$$

with the same $\mu_k(\sigma)$ as in (1).

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We shall restrict ourselves to the case where X is non-singular. Put $D_i = [V(\sigma_i)]$ for the one dimensional cone σ_i, and let $x_i \in H^2(X)$ denotes the Poincaré dual of D_i. The divisor D is written in the form $D = \sum_i d_i D_i$ with positive integers d_i. Put $\xi = \sum_i d_i x_i$. It is known that

$$a_k(P) = \int_X e^{\xi} \mathscr{T}^k(X) \ \text{ and } \ \operatorname{vol} P_\sigma = \int_X e^{\xi} x_\sigma,$$

where $\mathscr{T}^k(X) \in H^{2k}(X)_{\mathbb{Q}} = H^{2k}(X) \otimes \mathbb{Q}$ is the k-th component of the Todd cohomology class, the Poincaré dual of $\mathscr{T}_{n-k}(X)$, and $x_\sigma \in H^{2k}(X)$ is the Poincaré dual of $[V(\sigma)]$. The cohomology class x_σ can also be written as $x_\sigma = \prod_j x_j$ where the product runs over such j that σ_j is an edge of σ. Then the equality (2) can be rewritten as

$$\int_X e^{\xi} \mathscr{T}^k(X) = \sum_{\dim \sigma = k} \mu_k(\sigma) \int_X e^{\xi} x_\sigma. \tag{3}$$

The reader is referred to [4] Section 5.3 for details and Note 17 there for references.

In his paper [10] Morelli gave an answer to Danilov's question. Let $\mathrm{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0$ denote the field of rational functions of degree 0 on the Grassmann manifold of $(n-k+1)$-planes in $N_{\mathbb{Q}}$. For a cone σ of dimension k in $N_{\mathbb{R}}$ he associates a rational function $\mu_k(\sigma) \in \mathrm{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0$. With this $\mu_k(\sigma)$, the right hand side of (1) belongs to

$$\mathrm{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} A_{n-k}(X)_{\mathbb{Q}},$$

and the equality (1) means that the rational function with values in $A_{n-k}(X)_{\mathbb{Q}}$ in the right hand side is in fact a constant function equal to $\mathscr{T}_{n-k}(X)$ in $A_{n-k}(X)_{\mathbb{Q}}$. In other words this means that

$$\sum_{\sigma \in \Delta_X, \dim \sigma = k} \mu_k(\sigma)(E)[V(\sigma)] = \mathscr{T}_{n-k}(X)$$

for any generic $(n-k+1)$-plane E in $N_{\mathbb{Q}}$.

Morelli gives an explicit formula for $\mu_k(\sigma)$ when the toric variety is non-singular using Baum-Bott's residue formula for singular foliations [1] applied to the action of $(\mathbb{C}^*)^n$ on X. He then shows that the function $\mu_k(\sigma)$ is additive with respect to non-singular subdivisions of the cone σ. This fact leads to (1) in its general form.

One can ask a similar question about general classes other than the Todd class whether it is possible to define $\mu(x, \sigma) \in \mathrm{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0$ for $x \in A_{n-k}(X)$ in a canonical way to satisfy

$$x = \sum_{\sigma \in \Delta_X, \dim \sigma = k} \mu(x, \sigma)[V(\sigma)]. \tag{4}$$

When X is non-singular one can expect that $\mu(x, \sigma)$ satisfies a formula analogous to (3)

$$\int_X e^{\xi} x = \sum_{\dim \sigma = k} \mu(x, \sigma) \int_X e^{\xi} x_\sigma \tag{5}$$

for any cohomology class $\xi = \sum_i d_i x_i$. In this sense the formula does not explicitly refer to convex polytopes. Fulton and Sturmfels [5] used Minkowski weights to describe intersection theory of toric varieties. For complete non-singular varieties or $\mathbb{Q}$-factorial varieties X the Minkowski weight $\gamma_x : H^{2(n-k)}(X) \to \mathbb{Q}$ corresponding to $x \in H^{2k}(X)$ is defined by $\gamma_x(y) = \int_X xy$. Thus, if the d_i are considered as variables in ξ, the formula (5) is considered as describing γ_x as a linear combination of the Minkowski weights of γ_{x_σ}.

The purpose of the present paper is to establish the formula (5) by showing an explicit formula for $\mu(x,\sigma)$ when X is a $\mathbb{Q}$-factorial complete toric variety, that is, when Δ is a complete simplicial fan. Moreover our proof is based on a simple combinatorial argument which can be generalized to the case of multi-fans introduced by Masuda [8]. Topologically the formula concerns equivariant cohomology classes on so-called torus orbifolds (see [6]). This would suggest that actions of compact tori equipped with some nice conditions admit topological residue formulas similar to Baum-Bott' formula.

In Section 2 the definition of $\mu(x,\sigma)$ will be given and main results will be stated. In topological language, Theorem 2.1 states that (5) holds for any complete $\mathbb{Q}$-factorial toric varieties X. In Corollary 2.2 the explicit formula μ_k for the Todd class $x = \mathscr{T}(X)$ is given, generalizing that of Morelli. In Corollary 2.3 it will be shown that (4) holds for a complete $\mathbb{Q}$-factorial toric varieties X.

Section 3 is devoted to the proof of Theorem 2.1 and Corollary 2.3. Corollary 2.2 will be proved in a generalized form in Section 5. The results in Section 2 are generalized to complete multi-fans in Section 5 after some generalities about multi-fans and multi-polytopes are introduced in Section 4. Theorem 5.1, Corollary 5.2 and Corollary 5.3 are the corresponding generalized results. Roughly speaking Theorem 5.1 and Corollary 5.2 hold for (compact) torus orbifolds in general whereas Corollary 5.3 holds for torus orbifolds whose cohomology ring is generated by its degree two part like complete $\mathbb{Q}$-factorial toric varieties.

2. Statement of main results

It is convenient to describe a fan Δ in a form suited for combinatorial manipulation. We assume that the fan Δ is simplicial, that is, every k-dimensional cone of Δ has exactly k edges (one dimensional cones). Let $\Sigma^{(1)}$ denote the set of one dimensional cones. Then the set of cones of Δ forms a simplicial complex Σ with vertex set $\Sigma^{(1)}$. The set of $(k-1)$-simplices, i.e. k-dimensional cones, will be denoted by $\Sigma^{(k)}$. The cone corresponding to a simplex $J \in \Sigma$ will be denoted by $C(J)$ and the fan Δ is denoted by $\Delta = (\Sigma, C)$. Note that $\Sigma^{(0)}$ consists of a unique element o corresponding to the empty set as a subset of $\Sigma^{(1)}$ and $C(o) = 0$. (In [6] Σ with $\Sigma^{(0)}$ added is called augmented simplicial set.)

For each $i \in \Sigma^{(1)}$ let v_i be the primitive vector in $N \cap C(i)$. We put $\mathscr{V} = \{v_i\}_{i\in\Sigma^{(1)}}$ and $\mathscr{V}_J = \{v_j \mid j \in J\}$ for $J \in \Sigma$. Let $N_{J,\mathscr{V}}$ be the sublattice generated by $\mathscr{V}_J$ and N_J the minimal primitive (saturated) sublattice containing $N_{J,\mathscr{V}}$. The quotient group $N_J/N_{J,\mathscr{V}}$ is denoted by H_J. If $\sigma = C(J)$, H_J is the multiplicity along the normal direction at a generic point of $V(\sigma)$ in the toric variety (orbifold) X_Δ corresponding to Δ. The fan Δ is non-singular if and only if $H_I = 0$ for all $I \in \Sigma^{(n)}$, $n = \operatorname{rank}\Delta$. In this case $H_J = 0$ for all $J \in \Delta$ since H_J is contained in H_I if $J \subset I$.

We denote the torus $N_{\mathbb{R}}/N$ by T. N can be identified with $\operatorname{Hom}(S^1, T)$, and the dual $M = N^*$ with $\operatorname{Hom}(T, S^1)$. Since $\operatorname{Hom}(T, S^1)$ is identified with $H^1(T) = H^2(BT) = H^2_T(pt)$, M is identified with $H^2_T(pt)$.

The Stanley-Reisner ring of the simplicial set Σ is denoted by $H^*_T(\Delta)$. It is the quotient ring of the polynomial ring $\mathbb{Z}[x_i \mid i \in \Sigma^{(1)}]$ by the ideal generated by $\{x_K = \prod_{i\in K} x_i \mid K \subset \Sigma^{(1)},\ K \notin \Sigma\}$. It is considered as a ring over $H^*_T(pt)$ (regarded as embedded in $H^*_T(\Delta)$) by the formula

$$u = \sum_{i\in\Sigma^{(1)}} \langle u, v_i\rangle x_i. \tag{6}$$

When Δ is the fan Δ_X associated to a complete $\mathbb{Q}$-factorial toric variety X, $H_T^*(\Delta_X)_{\mathbb{Q}} = H_T^*(\Delta_X) \otimes \mathbb{Q}$ can be identified with the equivariant cohomology ring of X with respect to the action of compact torus T acting on X (see [8]).

For each $J \in \Sigma$ let $\{u_j^J\}$ be the basis of $N_{J,\mathscr{V}}^*$. $N_{J,\mathscr{V}}^*$ contains N_J^*. In particular $N_{I,\mathscr{V}}^*$ contains $N^* = M$ for $I \in \Sigma^{(n)}$ and will be considered as embedded in $M_{\mathbb{Q}} = H_T^2(pt)_{\mathbb{Q}}$. Define $\iota_I^* : H_T^2(\Delta)_{\mathbb{Q}} \to M_{\mathbb{Q}}$ by

$$\iota_I^*\left(\sum_{i\in\Sigma^{(1)}} d_i x_i\right) = \sum_{i\in I} d_i u_i^I. \tag{7}$$

ι_I^* extends to $H_T^*(\Delta)_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$. It is an $H_T^*(pt)_{\mathbb{Q}}$-module map, since

$$\iota_I^*(u) = u \text{ for } u \in H_T^*(pt)_{\mathbb{Q}}.$$

Let S be the multiplicative set in $H_T^*(pt)_{\mathbb{Q}}$ generated by non-zero elements in $H_T^2(pt)_{\mathbb{Q}}$. The push-forward $\pi_* : H_T^*(\Delta)_{\mathbb{Q}} \to S^{-1}H_T^*(pt)_{\mathbb{Q}}$ is defined by

$$\pi_*(x) = \sum_{I\in\Sigma^{(n)}} \frac{\iota_I^*(x)}{|H_I|\prod_{i\in I} u_i^I}. \tag{8}$$

It is an $H_T^*(pt)_{\mathbb{Q}}$-module map, and lowers the degrees by $2n$. It is known [6] that, if Δ is a complete simplicial fan, then the image of π_* lies in $H_T^*(pt)_{\mathbb{Q}}$.

Assume that Δ is complete. Let $p_* : H_T^*(\Delta)_{\mathbb{Q}} \to \mathbb{Q}$ be the composition of $\pi_* : H_T^*(\Delta)_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$ and $H_T^*(pt)_{\mathbb{Q}} \to H_T^0(pt)_{\mathbb{Q}} = \mathbb{Q}$. Note that p_* induces $\int_\Delta : H^*(\Delta)_{\mathbb{Q}} \to \mathbb{Q}$ as noted in [6] where $H^*(\Delta)_{\mathbb{Q}}$ is the quotient of $H_T^*(\Delta)_{\mathbb{Q}}$ by the ideal generated by $H_T^+(pt)_{\mathbb{Q}}$. Note that $H^*(\Delta)_{\mathbb{Q}}$ is defined independently of $\mathscr{V}$. If $\bar{x}$ denotes the image of $x \in H_T^*(\Delta)_{\mathbb{Q}}$ in $H^*(\Delta)_{\mathbb{Q}}$, then $\int_\Delta \bar{x} = p_*(x)$.

If $X = X_\Delta$ is the complete toric variety associated to Δ, then $H^*(\Delta)_{\mathbb{Q}}$ is identified with $H^*(X)_{\mathbb{Q}}$, π_* with the push-forward $H_T^*(X)_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$ and $\int_\Delta$ with the ordinary integral $\int_X$ (see [6]).

Assume that $1 \leq k$. For $J \in \Sigma^{(k)}$ let M_J be the annihilator of N_J and $\omega_J \in \bigwedge^{n-k} M$ be the element determined by M_J with an orentaiton, namely $\omega_J = u_1 \wedge \ldots \wedge u_{n-k}$ with an oriented basis $\{u_1, \ldots, u_{n-k}\}$ of M_J. Define $f^J(x_i) \in \bigwedge^{n-k+1} M_{\mathbb{Q}}$ by

$$f^J(x_i) = \iota_I^*(x_i) \wedge \omega_J \quad \text{with } J \subset I \in \Sigma^{(n)}.$$

$f^J(x_i)$ is well-defined independently of I containing J. Let $S^*(\bigwedge^{n-k+1} M_{\mathbb{Q}})$ be the symmetric algebra over $\bigwedge^{n-k+1} M_{\mathbb{Q}}$. $f^J : H_T^2(\Delta)_{\mathbb{Q}} \to \bigwedge^{n-k+1} M_{\mathbb{Q}}$ extends to $f^J : H_T^*(\Delta)_{\mathbb{Q}} \to S^*(\bigwedge^{n-k+1} M_{\mathbb{Q}})$. For $x = \prod_i x_i^{\alpha_i} \in H_T^{2k}(\Delta)_{\mathbb{Q}}$ we put

$$f^J(x) = (f^J(x_i))^{\alpha_i}.$$

The definition of f^J depends on the orientations chosen, but $\frac{f^J(x)}{f^J(x_J)}$ does not. It belongs to the fraction field of the symmetric algebra $S^*(\bigwedge^{n-k+1} M_{\mathbb{Q}})$ and has degree 0. Hence it can be considered as an element of $Rat(\mathbb{P}(\bigwedge^{n-k+1} N_{\mathbb{Q}}))_0$, the field of rataional functions of degree 0 on $\mathbb{P}(\bigwedge^{n-k+1} N_{\mathbb{Q}})$. Let $\nu^* : Rat(\mathbb{P}(\bigwedge^{n-k+1} N_{\mathbb{Q}}))_0 \to Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ be the induced homomorphism of the Plücker embedding $\nu : G_{n-k+1}(N_{\mathbb{Q}}) \to \mathbb{P}(\bigwedge^{n-k+1} N_{\mathbb{Q}})$. The image $\nu^*(\frac{f^J(x)}{f^J(x_J)})$ will be denoted by $\mu(x, J)$.

Our first main result is stated in the following

Theorem 2.1. *Let Δ be a complete simplicial fan in a lattice N of rank n and $x \in H_T^{2k}(\Delta)_{\mathbb{Q}}$. For any $\xi \in H_T^2(\Delta)_{\mathbb{Q}}$ we have*

$$p_*(e^{\xi}x) = \sum_{J\in\Sigma^{(k)}} \mu(x,J)p_*(e^{\xi}x_J) \quad \text{in } Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0. \tag{9}$$

We say that $\xi = \sum_i d_i x_i \in H_T^2(\Delta)$ is T-Cartier if $\iota_I^*(\xi)$ belongs to $M = H_T^2(pt)$ for all $I \in \Sigma^{(n)}$. When Δ and ξ come from a convex lattice polytope P, ξ is T-Cartier if and only if P is a lattice polytope. In this case we have

$$p_*(e^{\xi}x_J) = \frac{\operatorname{vol} P_J}{|H_J|},$$

where P_J is the face of P corresponding to J ($P_J = P_\sigma$ with $\sigma = C(J)$), cf. e.g. [4]. Furthermore it will be shown in Section 4 that there is an element $\mathscr{T}_T(\Delta) \in H_T^*(\Delta)_{\mathbb{Q}}$ such that

$$p_*(e^{\nu\xi}\mathscr{T}_T(\Delta)) = \#(\nu P) = \sum_{k=0}^{n} a_k(P)\nu^{n-k}.$$

Applying Theorem 2.1 to $x = \mathscr{T}_T(\Delta)_k$ and the above ξ the following Corollary will be obtained generalizing Morelli's formula.

Corollary 2.2. *Let P be a convex simple lattice polytope, Δ the associated complete simplicial fan. Then we have*

$$a_k(P) = \sum_{J\in\Sigma^{(k)}} \mu_k(J) \operatorname{vol} P_J$$

with

$$\mu_k(J) = \frac{1}{|H_J|}\sum_{h\in H_J} \nu^* \left(\prod_{j\in J} \frac{1}{1-\chi(u_j^J,h)e^{-f^J(x_j)}} \right)_0$$

in $\mathrm{Rat}(G_{n-k+1}(N_{\mathbb{C}}))_0$*, where* $\chi(u,h) = e^{2\pi\sqrt{-1}\langle u, v(h)\rangle}$ *for* $u \in N_{J,\mathscr{V}}^*$ *and* $v(h) \in N_J$ *is a lift of* $h \in H_J$ *to* N_J*.*

As another immadiate corollay of Theorem 2.1 we obtain

Corollary 2.3. *Let Δ be a complete simplicial fan and $x \in H_T^{2k}(\Delta)_{\mathbb{Q}}$. Then*

$$\bar{x} = \sum_{J\in\Sigma^{(k)}} \mu(x,J)\bar{x}_J \quad \text{in } Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} H_T^{2k}(\Delta)_{\mathbb{Q}}.$$

Proofs of Theorem 2.1 and Corollary 2.3 will be given in the next section.

3. Proof of Theorem 2.1 and Corollary 2.3

For a primitive sublattice E of N of rank $n-k+1$ let $w_E \in \bigwedge^{n-k+1} N$ be a representative of $\nu(E) \in \mathbb{P}(\bigwedge^{n-k+1} N_{\mathbb{Q}})$. The equality (9) is equivalent to the condition that

$$p_*(e^{\xi}x) = \sum_{J\in\Sigma^{(k)}} \frac{f^J(x)}{f^J(x_J)}(w_E)p_*(e^{\xi}x_J) \quad \text{holds for every generic } E.$$

Let E be a generic primitive sublattice in N of rank $n-k+1$. The intersection $E\cap N_J$ has rank one for each $J \in \Sigma^{(k)}$. Take a non-zero vector $v_{E,J}$ in $E \cap N_J$. (One can choose $v_{E,J}$ to be the unique primitive vector contained in $E \cap C(J)$. But any non-zero vector will suffice for the later use.) For $x \in H_T^{2k}(\Delta)$ and $J \in \Sigma^{(k)}$ the value of $\iota_I^*(x)$ evaluated

on $v_{E,J}$ for $I \in \Sigma^{(n)}$ containing J depends only on $\iota_J^*(x)$ so that it will be denoted by $\iota_J^*(x)(v_{E,J})$. Similarly we shall simply write $\langle u_j^J, v_{E,J}\rangle$ instead of $\langle u_j^I, v_{E,J}\rangle$.

Lemma 3.1. *Put $f_j^J = u_j^J \wedge \omega_J$. Then*

$$a\langle f_j^J, w_E\rangle = \langle u_j^J, v_{E,J}\rangle,$$

where a is a non-zero constant depending only on $v_{E,J}$.

Proof. Take an oriented basis $u_1, \dots, u_{n-k}$ of M_J. Take also a basis $w_1, \dots, w_{n-k+1}$ of E and write $v_{E,J} = \sum_l c_l w_l$. Then, since $\langle u_i, v_{E,J}\rangle = 0$,

$$\sum_{l=1}^{n-k+1} c_l \langle u_i, w_l\rangle = 0, \quad \text{for } i = 1, \dots, n-k.$$

The matrix $(a_{il}) = (\langle u_i, w_l\rangle)$ has rank $n-k$ and we get

$$(c_1, \dots, c_{n-k+1}) = a(A_1, \dots, A_{n-k+1}), \quad a \neq 0,$$

where

$$A_l = (-1)^{l-1} \det \begin{pmatrix} a_{11} & \dots & \widehat{a_{1l}} & \dots & a_{1\,n-k+1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-k\,1} & \dots & \widehat{a_{n-k\,l}} & \dots & a_{n-k\,n-k+1} \end{pmatrix}.$$

Then

$$\begin{aligned} \langle u_j^J, v_{E,J}\rangle &= \sum_{l=1}^{n-k+1} c_l \langle u_j^J, w_l\rangle \\ &= a \sum_{l=1}^{n-k+1} \langle u_j^J, w_l\rangle A_l \\ &= a \det \begin{pmatrix} \langle u_j^J, w_1\rangle & \dots & \langle u_j^J, w_{n-k+1}\rangle \\ \langle u_1, w_1\rangle & \dots & \langle u_1, w_{n-k+1}\rangle \\ \dots & \dots & \dots \\ \langle u_{n-k}, w_1\rangle & \dots & \langle u_{n-k}, w_{n-k+1}\rangle \end{pmatrix} \\ &= a\langle f_j^J, w_E\rangle \end{aligned}$$

where $f_j^J = u_j^J \wedge u_1 \cdots \wedge u_{n-k}$ and $w_E = w_1 \wedge \cdots \wedge w_{n-k+1}$. $\square$

Remark 3.1. Let X be a non-singular complete toric variety of dimension n and Δ the associated fan. Let $T = T^n$ be the compact torus acting on X. $E \cap N_J$ determines a subcircle $T^1_{E,J}$ of T. Then $T^1_{E,J}$ pointwise fixes an invariant complex submanifold X_J. Hence it acts on the normal vector space at each generic point in X_J. Then the numbers $\langle u_j^J, v_{E,J}\rangle$ are weights of this action.

Lemma 3.1 implies that

$$\frac{f^J(x)}{f^J(x_J)}(w_E) = \frac{\iota_J^*(x)}{\prod_{j\in J} u_j^J}(v_{E,J}).$$

Then the equality (9) in Theorem holds if and only if

$$p_*(e^\xi x) = \sum_{J\in\Sigma^{(k)}} \frac{\iota_J^*(x)}{\prod_{j\in J} u_j^J}(v_{E,J}) p_*(e^\xi x_J) \tag{10}$$

holds for every generic E.

The following lemma is easy to prove, cf. e.g. [6] Lemma 8.1.

Lemma 3.2. *The vector space $H_T^{2k}(\Delta)_{\mathbb{Q}}$ is spanned by elements of the form*

$$u_1 \cdots u_{k_1} x_{J_{k_1}},\ J_{k_1} \in \Sigma^{(k-k_1)},\ u_i \in M_{\mathbb{Q}},$$

with $0 \leq k_1 \leq k-1$.

Note. For $x = u_1 \cdots u_{k_1} x_{J_{k_1}},\ J_{k_1} \in \Sigma^{(k-k_1)}$, with $k_1 \geq 1$,

$$p_*(e^{\xi} x) = 0.$$

In view of this lemma we proceed by induction on k_1.

For $x = x_{J_0}$ with $J_0 \in \Sigma^{(k)}$, the left hand side of (10) is equal to $p_*(e^{\xi} x_{J_0})$. Since $i_J^*(x) = 0$ unless $J = J_0$ and $i_J^*(x)/\prod_{j\in J} u_j^J = 1$ for $J = J_0$, the right hand side is also equal to $p_*(e^{\xi} x_{J_0})$. Hence (10) holds with x of the form $x = x_{J_0}$ for $J_0 \in \Sigma^{(k)}$.

Assuming that (10) holds for x of the form $u_1 \cdots u_{k_1} x_{J_{k_1}}$ with $J_{k_1} \in \Sigma^{(k-k_1)}$, we shall prove that it also holds for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ with $J_{k_1+1} \in \Sigma^{(k-(k_1+1))}$. Put $K = J_{k_1+1}$.

Case a). u_{k_1+1} belongs to $M_{K\mathbb{Q}}$, that is, $\langle u_{k_1+1}, v_i \rangle = 0$ for all $i \in K$. In this case

$$u_{k_1+1} = \sum_{i \in \Sigma^{(1)} \setminus K} \langle u_{k_1+1}, v_i \rangle x_i$$

since $\langle u_{k_1+1}, v_i \rangle = 0$ for all $i \in K$. For $i \notin K$, $x_i x_{J_{k_1+1}}$ is either of the form x_{J^i} with $J^i \in \Sigma^{(k-k_1)}$ or equal to 0. Thus, for $x = u_1 \cdots u_{k_1} x_i x_{J_{k_1+1}}$ with $i \notin K$, the equality (10) holds by induction assumption, and it also holds for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ by linearity.

Case b). General case. We need the following

Lemma 3.3. *For $K \in \Sigma^{(k-k_1)}$ with $k_1 \geq 1$, the composition homomorphism $M_{K\mathbb{Q}} \subset M_{\mathbb{Q}} \to E_{\mathbb{Q}}^*$ is surjective.*

The proof will be given later. By this lemma, there exists an element $u \in M_{K\mathbb{Q}}$ such that

$$\langle u_{k_1+1}, v_{E,J} \rangle = \langle u, v_{E,J} \rangle \text{ for all } J \in \Sigma^{(k)}.$$

Note that $\langle \iota_J^*(u), v_{E,J} \rangle = \langle u, v_{E,J} \rangle$ for any $u \in M_{\mathbb{Q}}$. Then, in (10) for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ with $J_{k_1+1} \in \Sigma^{(k-(k_1+1))}$, we have

$$\begin{aligned} \iota_J^*(x)(v_{E,J}) &= \left(\prod_{i=1}^{k_1} \langle u_i, v_{E,J} \rangle\right) \langle u_{k_1+1}, v_{E,J} \rangle \\ &= \left(\prod_{i=1}^{k_1} \langle u_i, v_{E,J} \rangle\right) \langle u, v_{E,J} \rangle. \end{aligned}$$

Hence if we put $x' = u_1 \cdots u_{k_1} u x_{J_{k_1+1}}$, the right hand side of (10) is equal to

$$\sum_{J \in \Sigma^{(k)}} \frac{\iota_J^*(x')}{\prod_{j\in J} u_j^J}(v_{E,J}) p_*(e^{\xi} x_J).$$

This last expression is equal to $p_*(e^{\xi} x')$ since x' belongs to Case a). Furthermore $p_*(e^{\xi} x') = 0$ and $p_*(e^{\xi} x) = 0$ by Note after Lemma 3.2. Thus both side of (10) for

$x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ are equal to 0. This completes the proof of Theorem 2.1 except for the proof of Lemma 3.3.

Proof of Lemma 3.3.

Take a simplex $I \in \Sigma^{(n)}$ which contains K and a simplex $K' \in \Sigma^{(k-1)}$ such that $K \subset K' \subset I$. Such a K' exists since $k - k_1 \leq k - 1$. Then there are exactly $n - k + 1$ simplices $J^1, \ldots, J^{n-k+1} \in \Sigma^{(k)}$ such that $K' \subset J^i \subset I$. It is easy to see that the vectors $v_{E,J^1}, \ldots, v_{E,J^{n-k+1}}$ are linearly independent so that they span $E_{\mathbb{Q}}$. Moreover $M_{K'\mathbb{Q}}$ detects these vectors, that is, $M_{K'\mathbb{Q}} \to M_{\mathbb{Q}} \to E^*_{\mathbb{Q}}$ is surjective. Since $M'_K \subset M_K \subset M$, $M_{K\mathbb{Q}} \to E^*_{\mathbb{Q}}$ is surjective.

Proof of Corollary 2.3.

Fix a generic sublattice E and put $x' = \sum_{J \in \Sigma^{(k)}} \mu(x, J) x_J$. Then

$$p_*(e^{\xi} x') = \sum_{J \in \Sigma^{(k)}} \mu(x, J) p_*(e^{\xi} x_J) = p_*(e^{\xi} x)$$

by Theorem 2.1. It follows that $p_*(e^{\xi}(x' - x)) = 0$. Thus, in order to prove Corollary 2.3, it suffices to show that $p_*(e^{\xi} y) = 0$, $\forall \xi \in H^2_T(\Delta)_{\mathbb{Q}}$, implies that y belongs to the ideal $\mathscr{J}$ generated by $H^2_T(pt)_{\mathbb{Q}}$. Since $H^*(\Delta)_{\mathbb{Q}} = H^*_T(\Delta)_{\mathbb{Q}}/\mathscr{J} \cong H^*(X_\Delta)_{\mathbb{Q}}$ is a Poincaré duality space generated by $H^2(\Delta)_{\mathbb{Q}}$, $p_*(e^{\xi} y) = 0$ implies that $p_*(y) = 0$, i.e. $y \in \mathscr{J}$.

4. Multi-fans and multi-polytopes

The notion of multi-fan and multi-polytope were introduced in [8]. In this article we shall be concerned only with simplicial multi-fans. See [8, 6, 7] for details.

Let N be a lattice of rank n. A *simplicial multi-fan* in N is a triple $\Delta = (\Sigma, C, w)$ where $\Sigma = \coprod_{k=0}^n \Sigma^{(k)}$ is an (augmented) simplicial complex, C is a map from $\Sigma^{(k)}$ into the set of k-dimensional strongly convex rational polyhedral cones in the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ for each k, and w is a map $\Sigma^{(n)} \to \mathbb{Z}$. $\Sigma^{(0)}$ consists of a single element $o =$ the empty set. (The definition in [8] and [6] requires additional restriction on w.) We assume that any $J \in \Sigma$ is contained in some $I \in \Sigma^{(n)}$ and $\Sigma^{(n)}$ is not empty.

The map C is required to satisfy the following condition; if $J \in \Sigma$ is a face of $I \in \Sigma$, then $C(J)$ is a face of $C(I)$, and for any I, the map C restricted on $\Sigma(I) = \{J \in \Sigma \mid J \subset I\}$ is an isomorphism of ordered sets onto the set of faces of $C(I)$. It follows that $C(I)$ is necessarily a simplicial cone and $C(o) = 0$. A simplicial fan is considered as a simplicial multi-fan such that the map C on Σ is injective and $w \equiv 1$.

For each $K \in \Sigma$ we set

$$\Sigma_K = \{J \in \Sigma \mid K \subset J\}.$$

It inherits the partial ordering from Σ and becomes a simplicial set where $\Sigma_K^{(j)} \subset \Sigma^{(j+|K|)}$. K is the unique element in $\Sigma_K^{(0)}$. Let N_K be the minimal primitive sublattice of N containing $N \cap C(K)$, and N^K the quotient lattice of N by N_K. For $J \in \Sigma_K$ we define $C_K(J)$ to be the cone $C(J)$ projected on $N^K \otimes \mathbb{R}$. We define a function

$$w : \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}$$

to be the restrictions of w to $\Sigma_K^{(n-|K|)}$. The triple $\Delta_K = (\Sigma_K, C_K, w)$ is a multi-fan in N^K and is called the *projected multi-fan* with respect to $K \in \Sigma$. For $K = o$, the projected multi-fan Δ_o is nothing but Δ itself.

A vector $v \in N_{\mathbb{R}}$ will be called *generic* if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimesnsion less than n. For a generic vector v we set $d_v = \sum_{v \in C(I)} w(I)$, where the sum is understood to be zero if there is no such I.

Definition. A simplicial multi-fan $\Delta = (\Sigma, C, w)$ is called *pre-complete* if the integer d_v is independent of generic vectors v. In this case this integer will be called the *degree* of Δ and will be denoted by $\deg(\Delta)$. It is also called the *Todd genus* of Δ and is denoted by $\mathrm{Td}[\Delta]$. A pre-complete multi-fan Δ is said to be *complete* if the projected multi-fan Δ_K is pre-complete for every $K \in \Sigma$.

A multi-fan is complete if and only if the projected multi-fan Δ_J is pre-complete for every $J \in \Sigma^{(n-1)}$.

Like a toric variety gives rise to a fan, a torus orbifold gives rise to a complete simplicial multi-fan, though this correspondence is not one to one. A torus orbifold is a closed oriented orbifold with an action of a torus of half the dimension of the orbifold itself with non-empty fixed point set and with some additional conditions on the isotopy groups (see [6]). Most typical non-toric examples are given in [3]. Cobordism invariants of torus orbifolds are encoded in the associated multi-fans.

In the sequel we shall often consider a set $\mathscr{V}$ consisting of non-zero edge vectors v_i for each $i \in \Sigma^{(1)}$ such that $v_i \in N \cap C(i)$. We do not require v_i to be primitive. This has meaning for torus orbifolds (see [6]). For any $K \in \Sigma$ put $\mathscr{V}_K = \{v_i\}_{i \in K}$. Let $N_{K,\mathscr{V}}$ be the sublattice of N_K generated by $\mathscr{V}_K$. The quotient group $N_K/N_{K,\mathscr{V}}$ is denoted by $H_{K,\mathscr{V}}$.

Let $\Delta = (\Sigma, C, w)$ be a simplicial multi-fan in a lattice N. We define the equivariant cohomology $H_T^*(\Delta)$ of a multi-fan Δ as the Stanley-Reisner ring of the simplicial complex Σ as in Section 1.

Let $\mathscr{V} = \{v_i\}_{i \in \Sigma^{(1)}}$ be a set of prescribed edge vectors as before. Let $\{u_i^J\}_{i \in K}$ be the basis of $N^*_{K,\mathscr{V}}$ dual to $\mathscr{V}_K$. We define a homomorphism $M = N^* = H_T^2(pt) \to H_T^2(\Delta)$ by the same formula (6) as in the case of fans. Since this definition depends on the set $\mathscr{V}$, the $H_T^*(pt)$-module structure of $H_T^*(\Delta)$ also depends on $\mathscr{V}$. To emphasize this fact we shall use the notation $H_T^*(\Delta, \mathscr{V})$. When all the v_i are taken primitive, the notation $H_T^*(\Delta)$ is used.

For $I \in \Sigma^{(n)}$ the map $\iota_I^* : H_T^*(\Delta, \mathscr{V})_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$ is defined by (7) as in the case of fans. On the other hand the definition of the push-forward is altered from (8) to

$$\pi_*(x) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)\iota_I^*(x)}{|H_I| \prod_{i \in I} u_i^I},$$

cf. [6]. If Δ is complete the image of π_* lies in $H_T^*(pt)_{\mathbb{Q}}$ as in the case of fans, and the map $p_* : H_T^*(\Delta, \mathscr{V})_{\mathbb{Q}} \to \mathbb{Q}$ is also defined in a similar way.

Let $K \in \Sigma^{(k)}$ and let $\Delta_K = (\Sigma_K, C_K, w_K)$ be the projected multi-fan. The link Lk K of K in Σ is a simplicial complex consisting of simplices J such that $K \cup J \in \Sigma$ and $K \cap J = \emptyset$. It will be denoted by Σ'_K in the sequel. There is an isomorphism from Σ'_K to Σ_K sending $J \in {\Sigma'}_K^{(l)}$ to $K \cup J \in \Sigma_K^{(l)}$. We consider the polynomial ring R_K generated by $\{x_i \mid i \in K \cup {\Sigma'}_K^{(1)}\}$ and the ideal $\mathscr{I}_K$ generated by monomials $x_J = \prod_{i \in J} x_i$ such that $J \notin \Sigma(K) * \Sigma'_K$ where $\Sigma(K) * \Sigma'_K$ is the join of $\Sigma(K)$ and Σ'_K. We define the equivariant cohomology $H_T^*(\Delta_K)$ of Δ_K with respect to the torus T as the quotient ring $R_K/\mathscr{I}_K$.

If $\mathscr{V}$ is a set of prescribed edge vectors, $H_T^2(pt)$ is regarded as a submodule of $H_T^2(\Delta_K)$ by a formula similar to (6). This defines an $H_T^*(pt)$-module structure on $H_T^*(\Delta_K)$ which will be denoted by $H_T^*(\Delta_K, \mathscr{V})$ to specify the dependence on $\mathscr{V}$. The projection $H_T^*(\Delta, \mathscr{V}) \to$

$H_T^*(\Delta_K, \mathscr{V})$ is defined by sending x_i to x_i for $i \in K \cup \Sigma'^{(1)}_K$ and putting $x_i = 0$ for $i \notin K \cup \Sigma'^{(1)}_K$. The restriction homomorphism $\iota_I^* : H_T^*(\Delta_K, \mathscr{V})_\mathbb{Q} \to H_T^*(pt)_\mathbb{Q}$ for $I \in \Sigma_K^{(n-k)}$ and the push-forward $\pi_* : H_T^*(\Delta_K, \mathscr{V})_\mathbb{Q} \to S^{-1}H_T^*(pt)_\mathbb{Q}$ are also defined in a similar way as before.

Given $\xi = \sum_{i \in K \cup \Sigma'^{(1)}_K} d_i x_i \in H_T^2(\Delta_K, \mathscr{V})_\mathbb{R}$, $d_i \in \mathbb{R}$, let A_K^* be the affine subspace in the space $M_\mathbb{R}$ defined by $\langle u, v_i \rangle = d_i$ for $i \in K$. Then we introduce a collection $\mathscr{F}_K = \{F_i \mid i \in \Sigma'^{(1)}_K\}$ of affine hyperplanes in A_K^* by setting

$$F_i = \{u \mid u \in A_K^*,\ \langle u, v_i \rangle = d_i\}.$$

The pair $\mathscr{P}_K(\xi) = (\Delta_K, \mathscr{F}_K)$ will be called a *multi-polytope* associated with ξ; see [7]. In case $K = o \in \Sigma^{(0)}$, $\mathscr{P}_K(\xi)$ is simply denoted by $\mathscr{P}(\xi)$.

For $\xi = \sum_{i \in \Sigma^{(1)}} d_i x_i$ and $K \in \Sigma^{(k)}$ put $\xi_K = \sum_{i \in K \cup \Sigma'^{(1)}_K} d_i x_i$ and $\mathscr{P}(\xi)_K = \mathscr{P}_K(\xi_K)$. It will be called the *face* of $\mathscr{P}(\xi)$ corresponding to K.

For $I \in \Sigma_K^{(n-k)}$, i.e. $I \in \Sigma^{(n)}$ with $I \supset K$, we put $u_I = \cap_{i \in I} F_i = \cap_{i \in I \setminus K} F_i \cap A_K^* \in A_K^*$. Note that u_I is equal to $\iota_I^*(\xi)$. The dual vector space $(N_\mathbb{R}^K)^*$ of $N_\mathbb{R}^K$ is canonically identified with the subspace $M_{K\mathbb{R}}$ of $M_\mathbb{R} = H_T^2(pt)_\mathbb{R}$. It is parallel to A_K^*, and u_i^I lies in $M_{K\mathbb{R}}$ for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. A vector $v \in N_\mathbb{R}^K$ is called generic if $\langle u_i^I, v \rangle \neq 0$ for any $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. The image in $N_\mathbb{R}^K$ of a generic vector in $N_\mathbb{R}$ is generic. We take a generic vector $v \in N_\mathbb{R}^K$, and define

$$(-1)^I := (-1)^{\#\{j \in I \setminus K \mid \langle u_j^I, v \rangle > 0\}} \quad \text{and} \quad (u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v \rangle > 0 \\ -u_i^I & \text{if } \langle u_i^I, v \rangle < 0. \end{cases}$$

for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. We denote by $C_K^*(I)^+$ the cone in A_K^* spanned by the $(u_i^I)^+$, $i \in I \setminus K$, with apex at u_I, and by ϕ_I its characteristic function. With these understood, we define a function $\mathrm{DH}_{\mathscr{P}_K(\xi)}$ on $A_K^* \setminus \cup_i F_i$ by

$$\mathrm{DH}_{\mathscr{P}_K(\xi)} = \sum_{I \in \Sigma_K^{(n-k)}} (-1)^I w(I) \phi_I.$$

As in [7] we call this function the *Duistermaat-Heckman function* associated with $\mathscr{P}_K(\xi)$. When $K = o$, $\mathrm{DH}_{\mathscr{P}(\xi)}$ is defined on $M_\mathbb{R} \setminus \cup_i F_i$.

Suppose that Δ is a simplicial fan. If all the d_i are positive and the set

$$P = \{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \leq d_i\}$$

is a convex polytope, then $\mathrm{DH}_{\mathscr{P}(\xi)}$ equals 1 on the interior of P and 0 on other components of $M_\mathbb{R} \setminus \cup_i F_i$.

The following theorem is fundamental in the sequel, cf. [7] Theorem 2.3 and [6] Corollary 7.4.

Theorem 4.1. *Let Δ be a complete simplicial multi-fan. Let $\xi = \sum_{i \in K \cup \Sigma'^{(1)}_K} d_i x_i \in H_T^2(\Delta_K, \mathscr{V})$ be as above with all d_i integers and put $\xi_+ = \sum_i (d_i + \epsilon) x_i$ with $0 < \epsilon < 1$. Then*

$$\sum_{u \in A_K^* \cap M} \mathrm{DH}_{\mathscr{P}_K(\xi_+)}(u) t^u = \sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|H_{I,\mathscr{V}}|} \sum_{h \in H_{I,\mathscr{V}}} \frac{\chi_I(\iota_I^*(\xi), h) t^{\iota_I^*(\xi)}}{\prod_{i \in I \setminus K} (1 - \chi_I(u_i^I, h)^{-1} t^{-u_i^I})}, \tag{11}$$

where $\chi_I(u, h)$ for $u \in N_{I,\mathscr{V}}^*$ is defined as in Corollary 2.2.

Note. The left hand side of (11) is considered as an element in the group ring of M over $\mathbb{R}$ or the character ring $R(T)\otimes\mathbb{R}$ considered as the Laurent polynomial ring in $t=(t_1,\dots,t_n)$. The equality shows that the right hand side, which is a rational fucnction of t, belongs to $R(T)$.

$\xi=\sum_i d_ix_i\in H_T^2(\Delta,\mathscr{V})$ is called T-Cartier if $\iota_I^*(\xi)\in M$ for all $I\in\Sigma^{(n)}$. This condition is equivalent to $u_I\in M$ for all $I\in\Sigma^{(n)}$. In this case $\mathscr{P}(\xi)$ is said lattice multi-polytope. If ξ is T-Cartier, then $\chi_I(\iota_I^*(\xi),h)\equiv 1$. Hence the above formula (11) for $\mathrm{DH}_{\mathscr{P}_K(\xi_{K+})}$ reduces in this case to

$$(12)\qquad \sum_{u\in A_K^*\cap M}\mathrm{DH}_{\mathscr{P}_K(\xi_{K+})}(u)t^u=\sum_{I\in\Sigma_K^{(n-k)}}\frac{w(I)}{|H_{I,\mathscr{V}}|}\sum_{h\in H_{I,\mathscr{V}}}\frac{t^{\iota_I^*(\xi_K)}}{\prod_{i\in I\setminus K}(1-\chi_I(u_i^I,h)^{-1}t^{-u_i^I})}.$$

Let $H_T^{**}(\ )$ denote the completed equivariant cohomology ring. The Chern character ch sends $R(T)\otimes\mathbb{R}$ to $H_T^{**}(pt)_{\mathbb{R}}$ by $\mathrm{ch}(t^u)=e^u$. The image of (12) by ch is given by

$$(13)\qquad \sum_{u\in A_K^*\cap M}\mathrm{DH}_{\mathscr{P}_K(\xi_{K+})}(u)e^u=\sum_{I\in\Sigma_K^{(n-k)}}\frac{w(I)}{|H_{I,\mathscr{V}}|}\sum_{h\in H_{I,\mathscr{V}}}\frac{e^{\iota_I^*(\xi_K)}}{\prod_{i\in I\setminus K}(1-\chi_I(u_i^I,h)^{-1}e^{-u_i^I})}.$$

Assume that $\xi=\sum_i d_ix_i\in H_T^2(\Delta,\mathscr{V})$ is T-Cartier. The number $\#(\mathscr{P}(\xi)_K)$ is defined by

$$\#(\mathscr{P}(\xi)_K)=\sum_{u\in A_K^*\cap M}\mathrm{DH}_{\mathscr{P}_K(\xi_{K+})}(u).$$

It is obtained from (13) by setting $u=0$, that is, it is equal to the image of (13) by $H_T^{**}(pt)_{\mathbb{R}}\to H_T^0(pt)_{\mathbb{Q}}$.

The equivariant Todd class $\mathscr{T}_T(\Delta,\mathscr{V})$ is defined in such a way that

$$\pi_*(e^{\xi}\mathscr{T}_T(\Delta,\mathscr{V}))=\sum_{u\in M}\mathrm{DH}_{\mathscr{P}(\xi_+)}(u)e^u$$

for ξ T-Cartier. In order to give the definition we need some notations.

For simplicity identify the set $\Sigma^{(1)}$ with $\{1,2,\dots,m\}$ and consider a homomorphism $\eta:\mathbb{R}^m=\mathbb{R}^{\Sigma^{(1)}}\to N_{\mathbb{R}}$ sending $\mathbf{a}=(a_1,a_2,\dots,a_m)$ to $\sum_{i\in\Sigma^{(1)}}a_iv_i$. For $K\in\Sigma^{(k)}$ we define

$$\tilde{G}_{K,\mathscr{V}}=\{\mathbf{a}\mid\eta(\mathbf{a})\in N\text{ and }a_j=0\text{ for }j\notin K\}$$

and define $G_{K,\mathscr{V}}$ to be the image of $\tilde{G}_{K,\mathscr{V}}$ in $\tilde{T}=\mathbb{R}^m/\mathbb{Z}^m$. It will be written G_K for simplicity. The homomorphism η restricted on $\tilde{G}_{K,\mathscr{V}}$ induces an isomorphism

$$\eta_K:G_K\cong H_{K,\mathscr{V}}\subset T=N_{\mathbb{R}}/N.$$

Put

$$G_\Delta=\bigcup_{I\in\Sigma^{(n)}}G_I\subset\tilde{T}\quad\text{and}\quad DG_\Delta=\bigcup_{I\in\Sigma^{(n)}}G_I\times G_I\subset G_\Delta\times G_\Delta.$$

Let $v(g)=\mathbf{a}=(a_1,a_2,\dots,a_m)\in\mathbb{R}^m$ be a representative of $g\in\tilde{T}$. The factor a_i will be denoted by $v_i(g)$. It is determined modulo integers. If $g\in G_I$, then $v_i(g)$ is necessarily a rational number. Define a homomorphism $\chi_i:\tilde{T}\to\mathbb{C}^*$ by

$$\chi_i(g)=e^{2\pi\sqrt{-1}v_i(g)}$$

Let $g\in G_I$ and $h=\eta_I(g)\in H_{I,\mathscr{V}}$. Then $\eta(v(g))\in N_I$ is a representative of h in N_I which will be denoted by $v(h)$. Then, for $g\in G_I$ and $i\in I$,

$$v_i(g)\equiv\langle u_i^I,v(h)\rangle\mod\mathbb{Z},$$

and
$$\chi_i(g) = e^{2\pi\sqrt{-1}\langle u_i^I, v(h)\rangle} = \chi_I(u_i^I, h).$$

Let Δ be a complete simplicial multi-fan. Define
$$\mathscr{T}_T(\Delta, \mathscr{V}) = \sum_{g\in G_\Delta} \prod_{i\in\Sigma^{(1)}} \frac{x_i}{1-\chi_i(g)e^{-x_i}} \in H_T^{**}(\Delta, \mathscr{V})_{\mathbb{Q}}.$$

Proposition 4.2. *Let Δ be a complete simplicial multi-fan. Assume that $\xi \in H_T^2(\Delta, \mathscr{V})$ is T-Cartier. Then*
$$\pi_*(e^\xi \mathscr{T}_T(\Delta, \mathscr{V})) = \sum_{u\in M} \mathrm{DH}_{\mathscr{P}(\xi_+)}(u)e^u.$$
Consequently
$$p_*(e^\xi \mathscr{T}_T(\Delta, \mathscr{V})) = \#(\mathscr{P}(\xi)).$$

Proof. (cf. [6] Section 8). Let $g \in G_\Delta$ and $I \in \Sigma^{(n)}$. If $g \notin G_I$, then there is an element $i \notin I$ such that $\chi_i(g) \neq 1$; so
$$\frac{x_i}{1-\chi_i(g)e^{-x_i}} = (1-\chi_i(g))^{-1}x_i + \text{higher degree terms}$$
for such i . Hence $i_I^*(\frac{x_i}{1-\chi_i(g)e^{-x_i}}) = 0$. Therefore, only elements g in G_I contribute to $\iota_I^*(\mathscr{T}_T(\Delta, \mathscr{V}))$. Now suppose $g \in G_I$. Then $\chi_i(g) = 1$ for $i \notin I$, so $\iota_I^*(\frac{x_i}{1-\chi_i(g)e^{-x_i}}) = 1$ for such i. Finally, since $\iota_I^*(x_i) = u_i^I$ for $i \in I$, we have
$$\iota_I^*(\mathscr{T}_T(\Delta, \mathscr{V})) = \sum_{g\in G_I} \prod_{i\in I} \frac{u_i^I}{1-\chi_i(g)e^{-u_i^I}}.$$
This together with (13) shows that
$$\begin{aligned}
\pi_*(e^\xi \mathscr{T}_T(\Delta, \mathscr{V})) =& \pi_* \left(e^\xi \sum_{g\in G_\Delta} \prod_{i=1}^m \frac{x_i}{1-\chi_i(g)e^{-x_i}} \right) \\
=& \sum_{I\in\Sigma^{(n)}} \frac{w(I)e^{\iota_I^*(\xi)}}{|H_{I,\mathscr{V}}|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\chi_i(g)e^{-u_i^I})} \\
=& \sum_{u\in M} \mathrm{DH}_{\mathscr{P}(\xi_+)}(u)e^u.
\end{aligned}$$
□

More generally, for $K \in \Sigma^{(k)}$, define $\mathscr{T}_T(\Delta, \mathscr{V})_K$ by
$$\mathscr{T}_T(\Delta, \mathscr{V})_K = \sum_{g\in G_{\Delta_K}} \prod_{i\in\Sigma_K'^{(1)}} \frac{x_i}{1-\chi_i(g)e^{-x_i}} \in H_T^{**}(\Delta, \mathscr{V})_{\mathbb{Q}}.$$
Then the same proof as for Propsition 4.2 yields

Proposition 4.3. *Let Δ be a complete simplicial multi-fan. Assume that $\xi \in H_T^2(\Delta, \mathscr{V})$ is T-Cartier. Then*
$$\pi_*(e^\xi x_K \mathscr{T}_T(\Delta, \mathscr{V})_K) = \sum_{u\in A_K^*\cap M} \mathrm{DH}_{\mathscr{P}_K(\xi_{K+})}(u)e^u.$$
for $K \in \Sigma^{(k)}$, where $x_K = \prod_{i\in K} x_i$. Consequently
$$p_*(e^\xi x_K \mathscr{T}_T(\Delta, \mathscr{V})_K) = \#(\mathscr{P}(\xi)_K).$$

The lattice $M\cap A_K^*$ defines a volume element dV_K on A_K^*. For $\xi=\sum_{i\in K\cup\Sigma_K'^{(1)}} d_i x_i\in H_T^2(\Delta_K,\mathscr{V})_{\mathbb{R}}$, the volume $\operatorname{vol}\mathscr{P}_K(\xi)$ of $\mathscr{P}_K(\xi)$ is defined by

$$\operatorname{vol}\mathscr{P}_K(\xi)=\int_{A_K^*}\mathrm{DH}_{\mathscr{P}_K(\xi)}\,dV_K^*.$$

Proposition 4.4. *For $\xi=\sum_{i\in\Sigma^{(1)}} d_i x_i\in H_T^2(\Delta,\mathscr{V})_{\mathbb{R}}$*

$$\frac{1}{|H_{K,\mathscr{V}}|}\operatorname{vol}\mathscr{P}(\xi)_K=p_*(e^{\xi}x_K).$$

Proof. We shall give a proof only for the case where ξ is T-Cartier. The general case can be reduced to this case, cf. [6], Lemma 8.6. By Proposition 4.3

$$\#(\mathscr{P}(\xi)_K)=p_*(e^{\xi}x_K\mathscr{T}_T(\Delta,\mathscr{V})_K).$$

The highest degree term with respect to $\{d_i\}$ in the right hand side is nothing but $\operatorname{vol}\mathscr{P}(\xi)_K$ and is equal to

$$p_*\left(\frac{\xi^{n-k}}{(n-k)!}x_K\right)\sum_{g\in G_{\Delta_K}}\left(\prod_{i\in\Sigma_K'^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)_0,$$

where the suffix 0 means taking 0-th degree term. But

$$\left(\prod_{i\in\Sigma_K'^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)_0=\begin{cases}1, & g\in G_K\\ 0, & g\notin G_K.\end{cases}$$

Hence

$$\operatorname{vol}\mathscr{P}(\xi)_K=|G_K|p_*\left(\frac{\xi^{n-k}}{(n-k)!}x_K\right)=|H_{K,\mathscr{V}}|p_*(e^{\xi}x_K).$$

□

5. Generalization

The definition of $f^J(x)$ for simplcial fans can be also applied for simplicial multi-fans. Consequently $\mu(x,J)\in Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ is also defined. Theorem 2.1 is generalized in the following form.

Theorem 5.1. *Let Δ be a complete simplicial multi-fan and $x\in H_T^{2k}(\Delta,\mathscr{V})_{\mathbb{Q}}$. For any $\xi\in H_T^2(\Delta)_{\mathbb{Q}}$ we have*

$$p_*(e^{\xi}x)=\sum_{J\in\Sigma^{(k)}}\mu(x,J)p_*(e^{\xi}x_J)\quad\text{in } Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0.$$

Lemma 3.1, Lemma 3.2, Lemma 3.3 all hold in this new setting. Hence the proofs in Section 2 literally apply to prove Theorem 5.1.

As to Corollary 2.2 its generalization takes the following form.

Corollary 5.2. *Let Δ be a complete simplicial multi-fan in a lattice of rank n. Assume that $\xi\in H_T^2(\Delta,\mathscr{V})$ is T-Cartier. Set*

$$\#(\mathscr{P}(\nu\xi))=\sum_{k=0}^{n}a_k(\xi)\nu^{n-k}.$$

Then we have

$$a_k(\xi) = \sum_{J\in\Sigma^{(k)}} \mu_k(J)\operatorname{vol}\mathscr{P}(\xi)_J$$

with

$$\mu_k(J) = \frac{1}{|H_{J,\mathscr{V}}|}\nu^*\left(\sum_{h\in H_{J,\mathscr{V}}}\prod_{j\in J}\frac{1}{1-\chi(u_j^J,h)e^{-f^J(x_j)}}\right)_0$$

in $\mathrm{Rat}(G_{n-k+1}(N_{\mathbb{C}}))_0$.

Note. It can be proved without difficulty that $\mu_k(J)$ does not depend on the choice of $\mathscr{V}$. Hence one has only consider the case where all the v_i are primitive.

Proof. By Proposition 4.3

$$\#(\mathscr{P}(\nu\xi)) = p_*(e^{\nu\xi}\mathscr{T}_T(\Delta,\mathscr{V})).$$

Put $x = (\mathscr{T}_T(\Delta,\mathscr{V}))_k \in H_T^{2k}(\Delta,\mathscr{V})_{\mathbb{Q}}$. By (9) which is valid under the assumption of Theorem 5.1 too and by Proposition 4.4

$$a_k(\xi) = \sum_{J\in\Sigma^{(k)}}\nu^*\left(\frac{f^J(x)}{f^J(x_J)}\right)\frac{\operatorname{vol}\mathscr{P}(\xi)_J}{|H_{J,\mathscr{V}}|}.$$

Thus it suffices to show that

$$\frac{f^J(x)}{f^J(x_J)} = \left(\sum_{h\in H_{J,\mathscr{V}}}\prod_{j\in J}\frac{1}{1-\chi(u_j^J,h)e^{-f^J(x_j)}}\right)_0,$$

or

$$f^J(x) = \left(\sum_{h\in H_{J,\mathscr{V}}}\prod_{j\in J}\frac{f^J(x_j)}{1-\chi(u_j^J,h)e^{-f^J(x_j)}}\right)_k.$$

Let $g\in G_\Delta$. If $g\notin G_J$, then there is an element $i\notin J$ such that $\chi_i(g)\neq 1$, and, for such i,

$$f^J(\frac{x_i}{1-\chi_i(g)e^{-x_i}}) = f^J((1-\chi_i(g))^{-1}x_i + \text{higher degree terms}) = 0,$$

since $f^J(x_i)=0$. Thus

$$f^J\left(\prod_{i\in\Sigma^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right) = 0$$

for $g\notin G_J$.

If $g\in G_J$, then $\chi_i(g)=1$ for $i\notin J$. Thus

$$f^J\left(\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right) = f^J(1+\frac{1}{2}x_i + \text{higher degree terms}) = 1$$

for $g\in G_J,\ i\notin J$. It follows that

$$f^J\left(\sum_{g\in G_\Delta}\prod_{i\in\Sigma^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right) = \sum_{g\in G_J}\prod_{i\in J}\frac{f^J(x_i)}{1-\chi_i(g)e^{-f^J(x_i)}}.$$

This implies

$$f^J(\mathscr{T}_T(\Delta,\mathscr{V})_k) = \left(\sum_{h\in H_{J,\mathscr{V}}} \prod_{j\in J} \frac{f^J(x_j)}{1-\chi_J(u_j^J,h)e^{-f^J(x_j)}} \right)_k .$$

□

As to Corollary 5.3 we need to put an additional condition on the multi-fan Δ. A simplicial complex Σ is said to be $\mathbb{Q}$-Gorenstein* if

$$\widetilde{H}_i(\mathrm{Lk}\ \ J)_{\mathbb{Q}} = \begin{cases} \mathbb{Q}, & i = \dim \mathrm{Lk}\ \ J \\ 0, & i < \dim \mathrm{Lk}\ \ J \end{cases}$$

for all $J \in \Sigma^{(k)}$, $\ 0 \le k \le n$. It is equivalent to say that the realization $|\Sigma|$ of Σ is a $\mathbb{Q}$-homology manifold and has the same $\mathbb{Q}$-homology with the sphere S^{n-1}. A complete simplicial multi-fan $\Delta = (\Sigma, C, w)$ is called $\mathbb{Q}$-Gorenstein* if Σ is $\mathbb{Q}$-Gorenstein*.

Corollary 5.3. *Let Δ be a $\mathbb{Q}$-Gorenstein* simplicial multi-fan and $x \in H_T^{2k}(\Delta)_{\mathbb{Q}}$. Then*

$$\bar{x} = \sum_{J\in\Sigma^{(k)}} \mu(x,J)\bar{x}_J \quad \textit{in } Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} H^{2k}(\Delta)_{\mathbb{Q}}.$$

We shall show that $H^*(\Delta)_{\mathbb{Q}}$ is a Poincaré duality space and is generated by $H^2(\Delta)_{\mathbb{Q}}$ if Δ is $\mathbb{Q}$-Gorenstein*. Then the proof of Corollary 2.3 can be applied in this case too.

Our construction follows [3]. Assume that Σ is $\mathbb{Q}$-Gorenstein*.

Let Σ^* be the dual complex of Σ. It is triangulated by the barycentric subdivision of Σ. The set of dual cells are in one to one corresondence with $\bigsqcup_{k=1}^n \Sigma^{(k)}$. The dual cell correponding to $K \in \Sigma^{(k)}$ is denoted by K^*.

Let P be the cone over Σ^*. P is itself a $\mathbb{Q}$-homology cell since Σ is $\mathbb{Q}$-Gorenstein*. It is considered as the dual cell o^* of o. The dual cell K^* of $K \in \Sigma^{(k)}$ has codimension k in P. For $p \in P$ define $D(p)$ to be the minimal dual cell containing p. The sublattice N_K determindes a subtorus T_K of T. We put $T_p = T_K$ when $D(p) = K^*$.

Then put $\tilde{P} = T \times P/\sim$ where the equivalence relation $\sim$ is defined by

$$(g,p) \sim (h,q) \Longleftrightarrow p = q,\ gh^{-1} \in T_p.$$

$\tilde{P}$ has a natural T-action. It is easy to see that $\tilde{P}$ is an orientable $\mathbb{Q}$-homology manifold.

Theorem 4.8 of [3] says that the cohomology ring $H_T^*(\tilde{P})_{\mathbb{Q}}$ is isomorphic to $H_T^*(\Delta)_{\mathbb{Q}}$. Theorem 5.10 of [3] tells us that $H^*(\tilde{P})_{\mathbb{Q}}$ is isomorphic to $H_T^*(\tilde{P})_{\mathbb{Q}}/\mathscr{J}$ where $\mathscr{J}$ is the ideal generated by the image of $H_T^*(pt)_{\mathbb{Q}}$ in $H_T^*(\tilde{P})_{\mathbb{Q}}$. It follows that $H^*(\Delta)_{\mathbb{Q}}$ is isomorphic to $H^*(\tilde{P})_{\mathbb{Q}}$. Since $\tilde{P}$ is an orientable $\mathbb{Q}$-homology manifold, $H^*(\Delta)_{\mathbb{Q}}$ is a Poincaré duality space generated by its degree two part.

This finishes the proof of Corollary 5.3.

Remark 5.1. Let X be a torus orbifold and Δ_X the associated multi-fan. It is known that, if the cohomology ring $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$, then $H^*(X)_{\mathbb{Q}}$ is isomorphic to $H^*(\Delta_X)_{\mathbb{Q}}$ ([8] Proposition 3.4) and Δ_X is $\mathbb{Q}$-Gorenstein* ([9] Lemma 8.2). Hence Corollary 5.3 holds for $x \in H_T(X)_{\mathbb{Q}}$ and $\bar{x} \in H^*(X)_{\mathbb{Q}}$.

Remark 5.2. When Δ is the fan associated to a convex polytope P and $\xi = D$, the Cartier divisor associated to P, we know (see, e.g. [4]) that

$$\mu_0(o) = 1,\ a_0(\xi) = \mathrm{vol}\,\mathscr{P}(\xi),\ \mu_1(i) = \frac{1}{2},\ a_1(\xi) = \frac{1}{2}\sum_{i\in\Sigma^{(1)}} \mathrm{vol}\,\mathscr{P}(\xi)_i.$$

This is also true for simplicial multi-fans and T-Cartier ξ.

As to a_n we have

$$a_n(\xi) = \mathrm{Td}(\Delta).$$

In fact $a_n(\xi) = p_*(\mathscr{T}_T(\Delta, \mathscr{V})) = (\pi_*(\mathscr{T}_T(\Delta, \mathscr{V})))_0$. Thus the above equality follows from the following rigidity property:

Theorem 5.4. *Let Δ be a complete simplicial multi-fan. Then*

$$\pi_*(\mathscr{T}_T(\Delta, \mathscr{V})) = (\pi_*(\mathscr{T}_T(\Delta, \mathscr{V})))_0 = \mathrm{Td}[\Delta].$$

See [6] Theorem 7.2 and its proof. Note that $\mathrm{Td}[\Delta] = 1$ for any complete simplicial fan Δ.

The explicit formula for $\pi_*(\mathscr{T}_T(\Delta, \mathscr{V}))$ is given by

$$\pi_*(\mathscr{T}_T(\Delta, \mathscr{V})) = \sum_{I\in\Sigma^{(n)}} \frac{w(I)}{|H_{I,\mathscr{V}}|} \sum_{h\in H_{I,\mathscr{V}}} \prod_{i\in I} \frac{1}{1-\chi_I(u_i^I, h)e^{-u_i^I}}.$$

This does not depend on the choice of $\mathscr{V}$ and is in fact equal to $\mathrm{Td}[\Delta]$.

Let Δ be a (not necessarily complete) simplicial fan in a lattice of rank n. Set

$$Td_T(\Delta) = \sum_{I\in\Sigma^{(n)}} \frac{1}{|H_I|} \sum_{h\in H_I} \prod_{i\in I} \frac{1}{1-\chi_I(u_i^I, h)e^{-u_i^I}} \in S^{-1}H_T^{**}(pt)_{\mathbb{Q}}.$$

For a simplex I let $\Sigma(I)$ be the simplicial complex consisting of all faces of I. For a fan $\Delta(I) = (\Sigma(I), C)$, $Td_T(\Delta(I))$ is denoted by $Td_T(I)$.

Theorem 5.5. *$Td_T(I)$ is additive with respect to simplicial subdivisions of the cone $C(I)$. Namely, if Δ is the fan determined by a simplicial subdivison of $C(I)$, then the following equality holds*

$$Td_T(\Delta) = Td_T(I).$$

For the proof it is sufficient to assume that $\Delta(I)$ and Δ are non-singular. In such a form a proof is give in [10]. The following corollary ensures that $\mu_k(J)$ can be defined for general polyhedral cones as pointed out by Morelli in [10].

Corollary 5.6. *Let $\Delta(J) = (\Sigma(J), C)$ be a fan in a lattice N of rank n where J is a simplex of dimension $k-1$. Then $\mu_k(J) \in \mathrm{Rat}(G_{n-k+1}(N_{\mathbb{Q}}))_0$ is additive with respect to simplicial subdivisions of $C(J)$.*

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